B.Sc- part-II(H), paper-III Curvature

27-07-2020

Q.1. Define curvature and radius of curvature

Soln: Curvature:→

The curvature K at any point of the curve, whose arcual distance from a fixed point on the curve is s and the inclination of the tangent at that point makes an angle ψ with the positive direction of the x-axis, is defined as

$$K = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}.$$

Radius of curvature:→

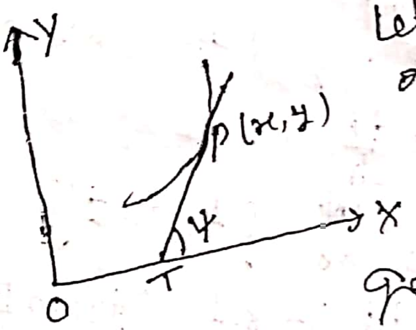
The radius of curvature at any point of a curve is defined as the reciprocal of curvature at that point.

If ρ be the radius of curvature at that point, then

$$\rho = \frac{1}{K} = \frac{ds}{d\psi}$$

Q.2. Find the radius of curvature for the Cartesian curve $y = f(x)$

Soln:



Let the Cartesian equation of the curve be $y = f(x)$.

We know that $\tan\psi = \frac{dy}{dx}$

Diff. with regard to x, we get,

$$\sec^2\psi \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow (1+\tan^2\psi)\frac{d\psi}{ds}\frac{ds}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow (1+\tan^2\psi)\frac{1}{\rho}\sec\psi = \frac{d^2y}{dx^2}$$

$$\Rightarrow (1+\tan^2\psi)\frac{1}{\rho}\left[\sqrt{1+\tan^2\psi}\right] = \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{1}{\rho}(1+\tan^2\psi)^{\frac{3}{2}} = \frac{d^2y}{dx^2} \Rightarrow \frac{1}{\rho}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \frac{d^2y}{dx^2}$$

Hence, $\rho = \dfrac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$, provided $\frac{d^2y}{dx^2} \neq 0$

$$= \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+p^2)^{\frac{3}{2}}}{q}$$

Proved.

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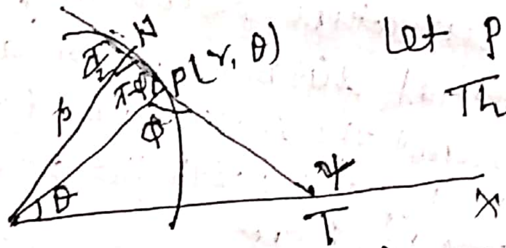
Q. 3 Prove that formula $\rho = r\frac{dr}{dp}$, where the symbols have their usual meaning.

Soln: Let P be any point (r, θ) on the curve. Then, $OP = r$, $\angle POX = \theta$



Draw the tangent PT to the curve at P.
Draw ON perpendicular to this tangent

Let $ON = p$, $\angle OPT = \phi$.

$\angle PTX = \psi$

$\angle OPN = \pi - \phi$,

We have $\sin\phi = r\frac{d\theta}{ds}$ and $\cos\phi = \frac{dr}{ds}$ —— (I)

Clearly $\psi = \theta + \phi$

Differentiating with respect to s, we get

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

i.e. $\frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$ ———— II

From the right-angled $\triangle ONP$, $\sin(\pi - \phi) = \frac{ON}{OP}$

i.e. $\sin\phi = \frac{p}{r} \Rightarrow p = r\sin\phi$

Differentiating both sides with regard to r, we get

$$\frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\cdot\frac{d\phi}{dr} \quad [\text{from I}]$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) = r\cdot\frac{1}{\rho} \quad [\text{from II}]$$

Hence, $\rho = r\frac{dr}{dp}$ Proved

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